Structural and Mechanical Systems Subjected to Constraints

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The characteristics of dynamic systems subjected to multiple linear constraints are determined by considering the constrained effects. Although there have been many researches to investigate the dynamic characteristics of constrained systems, most of them depend on numerical analysis like Lagrange multipliers method. In 1992, Udwadia and Kalaba presented an explicit form to describe the motion for constrained discrete systems. Starting from the method, this study determines the dynamic characteristics of the systems to have positive semidefinite mass matrix and the continuous systems. And this study presents a closed form to calculate frequency response matrix for constrained systems subjected to harmonic forces. The proposed methods that do not depend on any numerical schemes take more generalized forms than other research results.

Key Words : Generalized Inverse, Constraint, Continuous System, Substructure

1. Introduction

It is not easy work to determine the dynamic characteristics as well as the motion of constrained structural or mechanical systems. There have been many researches (Park et al., 1997 ; Park et al., 2000 ; Zheng et al. 1999) related to the constrained motion of structural or mechanical systems. In spite of a lot of efforts, there have been rarely closed methods to determine the constrained motion. Udwadia and Kalaba (1992) proposed an equation of motion for constrained systems. This method has advantages to be able to determine the constrained motion and constraint forces without depending on any numerical schemes. This method can apply to constrained discrete systems to have the positive definite mass matrix and its validity has been investigated through the applications of various control fields.

The frequency response function is a function to relate the input and output. A harmonic forcing function is utilized as the input. The frequency response function of unconstrained system is determined by substituting the generalized displacements, velocities, and accelerations of exponential form into the equation of motion for unconstrained system. However, if the system is subjected to multiple linear constraints that are functions of generalized coordinates to restrict the dynamic motion, the dynamic characteristics of the system shall be changed. Using harmonical solutions into Lagrange's equations formalism in

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connection with Lagrange's multipliers, Gurgoze (2000) provided an explicit equation to determine the frequency response matrix for constrained systems excited by harmonic forces. The constraints considered in his study were only limited to homogeneous functions of relative displacements and did not include interdependent state variables. As an extension of this work, he and his co-workers (Gurgoze, 1999 ; Gurgoze and Hizal, 2000 ; Gurgoze and Erol, 2002), presented equations to determine the eigenvalues and eigenvectors of constrained systems. However, these methods and applications require complicated intermediate procedures like the calculation of mode shapes of unconstrained systems.

Starting from the generalized inverse method provided by Udwadia and Kalaba, the aim of this study is to present simple and generalized forms for the dynamic characteristics of constrained systems. The derived equation gives more generalized forms than other methods without utilizing any numerical schemes. Also, modifying the vibration equation of the generalized inverse method, this study determines the dynamic characteristics of the systems to have positive semidefinite mass matrix and the continuous systems. The validity of the proposed method is illustrated by simple applications.

2. Generalized Inverse Method

Consider an unconstrained system of n particles whose configuration at time t is described in terms of an n-vector. The masses m_i, $i=1, 2, \cdots, n$ of the n particles will be taken to be constants. The equation of motion for such a system at time t may then be written, using the Lagrange or the Newtonian approach, as

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{C}\dot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{P} \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are positive definite mass, damping, and stiffness matrices, respectively. $\mathbf{P}$ is the forcing vector.

Let the system be subjected to the following m linear constraint equations of the form

$$\phi_i(\mathbf{q}, t)=0, \; i=1, 2, \cdots, m, \; m<n \tag{2}$$

Differentiating equation (2) with respect to time t, it can be written in matrix form

$$\mathbf{A}\ddot{\mathbf{q}}=\mathbf{b} \tag{3}$$

where $\mathbf{A}$ is known $m\times n$ matrix to be constants and $\mathbf{b}$ is $m\times 1$ vector.

The presence of the constraint set (2) brings into play forces of constraints, $\mathbf{F}^c$ so that the equation of motion at time t of the constrained system can be expressed as

$$\mathbf{M}\ddot{\mathbf{q}}=\mathbf{F}(\mathbf{q}, \dot{\mathbf{q}}, t)+\mathbf{F}^c(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{4}$$

where $\mathbf{q}$, $\dot{\mathbf{q}}$, and $\ddot{\mathbf{q}}$ refer to the n-displacement, velocity, and acceleration vectors, respectively, at time t of the constrained system that has been described above. Based on Gauss's principle (1829) and fundamental linear algebra, Udwadia and Kalaba derived the equations of motion for constrained systems written by

$$\ddot{\mathbf{q}}=\mathbf{a}+\mathbf{M}^{-1/2}(\mathbf{A}\mathbf{M}^{-1/2})^{+}(\mathbf{b}-\mathbf{A}\mathbf{a}) \tag{5}$$

where $\mathbf{a}=-\mathbf{M}^{-1}(\mathbf{C}\dot{\mathbf{q}}+\mathbf{K}\mathbf{q}-\mathbf{P})$ and '+' denotes Moore-Penrose inverse matrix.

The uniqueness and effectiveness of the generalized inverse method expressed by equation (5) have been verified by various kinds of applications. The generalized inverse method was derived under the fundamental assumption of positive definite mass matrix. Accordingly, the motion of constrained systems with positive semidefinite mass matrix cannot be described by equation (5) and the equation should be modified.

Let us consider a dynamic system that the mass matrix is positive semidefinite matrix of rank r, $r<n$, written as

$$\mathbf{M}=\begin{bmatrix} [\mathbf{M}_a]_{r\times r} & \mathbf{0}_{r\times s} \\ \mathbf{0}_{s\times r} & \mathbf{0}_{s\times s} \end{bmatrix}, \; r+s=n \tag{6}$$

where $\mathbf{0}$ denotes zero matrix. Because the inverse of the mass matrix cannot be calculated, the constrained motion from equation (5) cannot be calculated. In order to utilize equation (5) to the system to have the mass matrix of rank r, the mass matrix needs to be modified as

$$\mathbf{M}=\mathbf{M}_d-\mathbf{M}_o=\begin{bmatrix} \mathbf{M}_a & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}-\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{7}$$

where $\mathbf{I}$ is $s\times s$ unit matrix.

Utilizing equation (7) into equation (1), the equation of motion is expressed as

$$\mathbf{M}_d\ddot{\mathbf{q}}+\mathbf{C}\dot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{F}+\mathbf{M}_o\ddot{\mathbf{q}} \tag{8}$$

Assuming that this system is subjected to the constraints (2) and utilizing equation (8) as the modified equation of unconstrained motion, it is written as

$$\begin{aligned}\ddot{\mathbf{q}}=&\mathbf{M}_d^{-1/2}(\mathbf{A}\mathbf{M}_d^{-1/2})^{+}\mathbf{b}\\&-\lfloor \mathbf{I}-\mathbf{M}_d^{-1/2}(\mathbf{A}\mathbf{M}_d^{-1/2})^{+}\mathbf{A}\rfloor\mathbf{M}_d^{-1}(\mathbf{C}\dot{\mathbf{q}}+\mathbf{K}\mathbf{q}-\mathbf{F}-\mathbf{M}_o\ddot{\mathbf{q}})\end{aligned} \tag{9}$$

Arranging equation (9), the equation of motion for constrained systems to have positive semi-definite mass matrix is expressed as

$$\begin{aligned}[\mathbf{I}-\mathbf{M}^*\mathbf{M}_o]\ddot{\mathbf{q}}=&\mathbf{M}_d^{-1/2}(\mathbf{A}\mathbf{M}_d^{-1/2})^{+}\mathbf{b}\\&-\mathbf{M}^*(\mathbf{C}\dot{\mathbf{q}}+\mathbf{K}\mathbf{q}-\mathbf{F})\end{aligned} \tag{10}$$

where $\mathbf{M}^*=\lfloor \mathbf{I}-\mathbf{M}_d^{-1/2}(\mathbf{A}\mathbf{M}_d^{-1/2})^{+}\mathbf{A}\rfloor\mathbf{M}_d^{-1}$. Equation (10) can be extensively applied to the process to synthesize partitioned substructures including rigid-body structures.

3. Frequency Response Matrix for Constrained Systems

The equation of motion for a mechanical system subjected to harmonic excitation can be expressed by

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{C}\dot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\bar{\mathbf{F}}e^{i\omega t} \tag{11}$$

where **M**, **C**, and **K** are $n\times n$ mass, damping, and stiffness matrices, respectively, ω denotes the forcing frequency, and $\bar{\mathbf{F}}$ is $n\times 1$ force vector. Substituting $\mathbf{q}(t)=\bar{\mathbf{q}}e^{i\omega t}$ into equation (11), it yields the relation

$$\bar{\mathbf{q}}=\mathbf{H}(\omega)\bar{\mathbf{F}} \tag{12}$$

between the constant part of the input and response vectors. The complex matrix

$$\mathbf{H}(\omega)=(-\omega^2\mathbf{M}+i\omega\mathbf{C}+\mathbf{K})^{-1} \tag{13}$$

is referred to the frequency response matrix or the receptance matrix.

Let us assume that the system is subjected to m constraints like equation (2) and the second derivatives with respect to time are expressed in matrix form

$$\mathbf{A}\ddot{\mathbf{q}}=\mathbf{b}e^{i\omega t}=\mathbf{R}\bar{\mathbf{F}}e^{i\omega t} \tag{14}$$

where **R** is $m\times n$ matrix to relate the forcing vector $\bar{\mathbf{F}}e^{i\omega t}$ and the constraint equation (14). Substituting equations (11) and (14) into equation (5) and arranging the result, it is derived as

$$\bar{\mathbf{q}}=\hat{\mathbf{H}}(\omega)\bar{\mathbf{F}} \tag{15}$$

where

$$\hat{\mathbf{H}}(\omega)=\mathbf{G}^{-1}\mathbf{D}$$

$$\mathbf{G}=\lfloor -\mathbf{H}(\omega)^{-1}+\mathbf{M}^{1/2}(\mathbf{A}\mathbf{M}^{-1/2})^{+}\mathbf{A}\mathbf{M}^{-1}(\mathbf{H}(\omega)^{-1}+\omega^2\mathbf{M})\rfloor$$

$$\mathbf{D}=\lfloor \mathbf{M}^{1/2}(\mathbf{A}\mathbf{M}^{-1/2})^{+}(\omega^2\mathbf{R}+\mathbf{A}\mathbf{M}^{-1})-\mathbf{I}\rfloor$$

The receptance matrix is an index to establish the relation of input and output of a dynamic system subjected to harmonic excitation. Although Gurgoze determined the frequency response matrix of constrained systems subjected to harmonic excitation, he considered the constraints of the form $\mathbf{A}\ddot{\mathbf{q}}=\mathbf{0}$ instead of equation (3). Thus, it seems that equation (15) gives a general form to express the frequency response matrix for constrained systems.

4. Eigenfrequency of Constrained Systems

For a dynamic system in a region Ω_p of Fig. 1(a) described by the generalized displacement vector $\mathbf{q}=[q_1\ q_2\ \cdots\ q_n]^T$, the free vibration equation is written in terms of matrix form

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{C}\dot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{0} \tag{16}$$

Assume that $(l+h)$ secondary systems composed of l systems of fixed-free end and h systems of free-free end are attached on the primary system as shown in Fig. 1(b). Separating the primary system and the secondary systems at their boundaries, ·the separated secondary systems exhibit two types of support conditions of free-free end and fixed-free end as shown in Fig. 1(c). The l secondary systems of fixed-free end have l DOF described by the displacement vector $\mathbf{z}_{fd}=[z^b_{1_{fd}}\ z^b_{2_{fd}}\ \cdots\ z^b_{l_{fd}}]^T$. The motion of h secondary systems of free-free end to possess rigid-body DOF is described by the displacement vector $\mathbf{z}_{fr}=[z^i_{1_{fr}}\ z^i_{1_{fr}}\ \cdots\ z^b_{(h-1)_{fd}}\ z^b_{h_{fd}}]^T$ of $2h$ DOF, where the

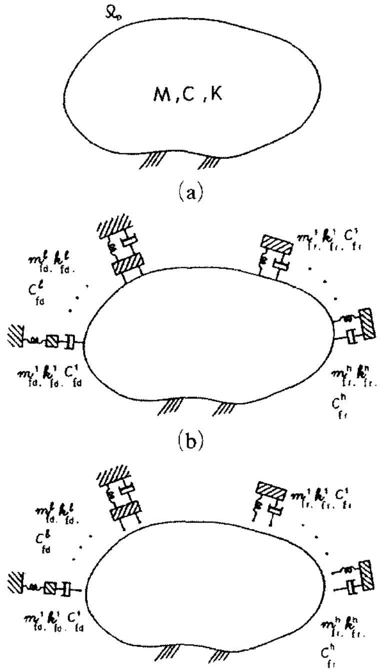

Fig. 1 Composition and decomposition of an entire structure and substructure; (a) an entire structure, (b) composition of an entire structure and substrucutres, (c) decomposition of primary structure and secondary structures

superscripts i and b indicate the inner and boundary sets, respectively. Accordingly, considering that the secondary systems of free-free end have zero-mass at the boundaries, the vibration equation of the secondary systems having $2h+l$ DOF can be written in terms of matrix form

$$\mathbf{M}_{sb}\ddot{\mathbf{z}}+\mathbf{C}_{sb}\dot{\mathbf{z}}+\mathbf{K}_{sb}\mathbf{z}=\mathbf{0},\ n\geq 2h+l \tag{17}$$

where

$$\mathbf{M}_{sb}=\begin{bmatrix}\mathbf{M}_{fd} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}^{i}_{fr} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{M}^{b}_{fr}\end{bmatrix},\ \mathbf{C}_{sb}=\begin{bmatrix}\mathbf{C}_{fd} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}^{i}_{fr} & \mathbf{C}^{ib}_{fr}\\ \mathbf{0} & \mathbf{C}^{bi}_{fr} & \mathbf{C}^{b}_{fr}\end{bmatrix}$$
$$\mathbf{K}_{sb}=\begin{bmatrix}\mathbf{K}_{fd} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}^{i}_{fr} & \mathbf{K}^{ib}_{fr}\\ \mathbf{0} & \mathbf{K}^{bi}_{fr} & \mathbf{K}^{b}_{fr}\end{bmatrix},\ \mathbf{M}_{sb}=\begin{bmatrix}\mathbf{z}_{fd}\\ \mathbf{z}^{i}_{fr}\\ \mathbf{z}^{b}_{fr}\end{bmatrix} \tag{18}$$

The subscripts fr and fd represent the substructures of free-free end and fixed-free end, respectively. The secondary systems of free-free end have zero-mass at the boundary DOF, hence in equation (18) $\mathbf{M}^{b}_{fr}=\mathbf{0}$. Also, the damping and stiffness matrices in equation (18) can be written by

$$\mathbf{C}^{i}_{fr}=\mathbf{C}^{b}_{fr}=-\mathbf{C}^{ib}_{fr}=-\mathbf{C}^{bi}_{fr} =diag\lfloor c_{1_{fr}}\ c_{2_{fr}}\cdots c_{h_{fr}}\rfloor \tag{19}$$

$$\mathbf{K}^{i}_{fr}=\mathbf{K}^{b}_{fr}=-\mathbf{K}^{ib}_{fr}=-\mathbf{K}^{bl}_{fr} =diag\lfloor k_{1_{fr}}\ k_{2_{fr}}\cdots k_{h_{fr}}\rfloor \tag{20}$$

In order to reduce the DOF of the primary system from n to g, $n>g\geq 1$, the transformation

$$\mathbf{q}=\boldsymbol{\Phi}_r\mathbf{y}_r \tag{21}$$

is utilized, where $\boldsymbol{\Phi}_r=\lfloor \varphi_1\ \varphi_2\cdots\varphi_g\rfloor$ is the modal matrix of the undamped system and $\mathbf{y}_r$ is the $\mathbf{g}\times 1$ modal displacement vector. Utilizing equation (21) into equation (16) with the assumption of classical damping matrix $\mathbf{C}$, it is written by

$$\ddot{\mathbf{y}}_r+\boldsymbol{\Psi}_r\dot{\mathbf{y}}_r+\boldsymbol{\Lambda}_r\mathbf{y}_r=\mathbf{0} \tag{22}$$

where

$$\boldsymbol{\Phi}_r^T\mathbf{M}\boldsymbol{\Phi}_r=\mathbf{I}_r,\ \boldsymbol{\Phi}_r^T\mathbf{C}\boldsymbol{\Phi}_r=\boldsymbol{\Psi}_r$$
$$\boldsymbol{\Phi}_r^T\mathbf{K}\boldsymbol{\Phi}_r=\boldsymbol{\Lambda}_r=diag(\omega_j^2),\ j=1,2,\cdots,g$$
$$\mathbf{y}_r=[y_1\ y_2\cdots y_g]^T$$

Based on equations (22) and (17), the vibration equation of the entire structure related to the displacement vector $\mathbf{u}_p=[\mathbf{y}_r{}^T\ \mathbf{z}^{b}_{fd}{}^T\ \mathbf{z}^{i}_{fr}{}^T\ \mathbf{z}^{b}_{fr}{}^T]^T$ can be expressed by

$$\mathbf{M}^c\ddot{\mathbf{u}}_p+\mathbf{C}^c\dot{\mathbf{u}}_p+\mathbf{K}^c\mathbf{u}_p=\mathbf{0} \tag{23}$$

where

$$\mathbf{M}^c=\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}_{sb}\end{bmatrix},\ \mathbf{C}^c=\begin{bmatrix}\boldsymbol{\Psi}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_{sb}\end{bmatrix},\ \mathbf{K}^c=\begin{bmatrix}\boldsymbol{\Lambda}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{K}_{sb}\end{bmatrix}$$

Considering common nodes between the systems at $(l+h)$ boundary points, the compatibility conditions

$$q_i=\sum_{o=1}^{g}\phi_{oi}y_o=z^{b}_{i_{fd}},\ i=1,2,\cdots,l \tag{24a}$$

$$q_j=\sum_{r=1}^{g}\phi_{rj}y_r=z^{b}_{s_{fr}},\ j=l+1,\ l+2,\cdots,\ l+h,\quad s=1,2,\cdots,h \tag{24b}$$

at the boundaries should be satisfied. The g generalized displacements of the primary system and the $(l+2h)$ displacements of the secondary systems are expressed by a displacement vector $\mathbf{u}_p$.

As shown in equation (5), the actual acceleration of constrained systems is expressed as the sum of the unconstrained acceleration and the additional acceleration due to the presence of constraints. The unconstrained accelerations of the entire system are determined by equation (23). The additional acceleration can be obtained in the following. The constraints of equation (24) can be expressed in matrix form

$$\mathbf{A}\mathbf{u}_{\mathrm{p}}=\mathbf{0} \tag{25}$$

where $\mathbf{A}$ is $(l+h)\times(g+l+2h)$ matrix to be constants written by

$$\mathrm{A}=\left[\begin{array}{cccc|cccc|cccc|cccc}
\multicolumn{4}{c}{\overbrace{\hphantom{\phi_{11}\ \phi_{21}\ \cdots\ \phi_{g1}}}^{g}} & \multicolumn{4}{c}{\overbrace{\hphantom{-1\ 0\ \cdots\ 0}}^{l}} & \multicolumn{4}{c}{\overbrace{\hphantom{0\ 0\ \cdots\ 0}}^{h}} & \multicolumn{4}{c}{\overbrace{\hphantom{0\ 0\ \cdots\ 0}}^{h}} \\
\phi_{11} & \phi_{21} & \cdots & \phi_{g1} & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\phi_{12} & \phi_{22} & \cdots & \phi_{g2} & 0 & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\phi_{1l} & \phi_{2l} & \cdots & \phi_{gl} & 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\phi_{1(l+1)} & \phi_{2(l+1)} & \cdots & \phi_{g(l+1)} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\
\phi_{1(l+2)} & \phi_{2(l+2)} & \cdots & \phi_{g(l+2)} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\phi_{1(l+h)} & \phi_{2(l+h)} & \cdots & \phi_{g(l+h)} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & -1
\end{array}\right] \tag{26}$$

Differentiating equation (25) twice with respect to time t, it follows that

$$\mathbf{A}\ddot{\mathbf{u}}_{\mathrm{p}}=\mathbf{0} \tag{27}$$

The additional acceleration due to constraints is obtained by utilizing equations (23) and (27) into the second term of equation (5). Consequently, substitution of equations (23) and (27) into equation (10) leads to the equations of motion for the entire system

$$(\mathbf{I}-\mathbf{M}^{*}\mathbf{M}_{0})\ddot{\mathbf{u}}_{\mathrm{p}}+\mathbf{M}^{*}\mathbf{C}\dot{\mathbf{u}}_{\mathrm{p}}+\mathbf{M}^{*}\mathbf{K}\mathbf{u}_{\mathrm{p}}=0 \tag{28}$$

where $\mathbf{M}^{*}=\lfloor \mathbf{I}-\mathbf{M}_{\mathrm{d}}^{-1/2}(\mathbf{A}\mathbf{M}_{\mathrm{d}}^{-1/2})^{+}\mathbf{A}\rfloor\ \mathbf{M}_{\mathrm{d}}^{-1}$

$$\mathbf{M}_{\mathrm{d}}=\begin{bmatrix} \mathbf{I} & 0 & 0 & 0 \\ 0 & \mathbf{M}_{\mathrm{fd}} & 0 & 0 \\ 0 & 0 & \mathbf{M}_{\mathrm{fr}}^{\mathrm{i}} & 0 \\ 0 & 0 & 0 & \mathbf{I} \end{bmatrix},\ \mathbf{M}_{\mathrm{o}}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{I} \end{bmatrix}$$
$$\mathbf{C}=\begin{bmatrix} \Psi_{\mathrm{r}} & 0 & 0 & 0 \\ 0 & \mathbf{C}_{\mathrm{fd}} & 0 & 0 \\ 0 & 0 & \mathbf{C}_{\mathrm{fr}}^{\mathrm{i}} & \mathbf{C}_{\mathrm{fr}}^{\mathrm{ib}} \\ 0 & 0 & \mathbf{C}_{\mathrm{fr}}^{\mathrm{bi}} & \mathbf{C}_{\mathrm{fr}}^{\mathrm{b}} \end{bmatrix},\ \mathbf{K}=\begin{bmatrix} \Lambda_{r} & 0 & 0 & 0 \\ 0 & \mathbf{K}_{\mathrm{fd}} & 0 & 0 \\ 0 & 0 & \mathbf{K}_{\mathrm{fr}}^{\mathrm{i}} & \mathbf{K}_{\mathrm{fr}}^{\mathrm{ib}} \\ 0 & 0 & \mathbf{K}_{\mathrm{fr}}^{\mathrm{bi}} & \mathbf{K}_{\mathrm{fr}}^{b} \end{bmatrix},\ \mathbf{u}_{\mathrm{p}}=\begin{bmatrix} \mathbf{y}_{\mathrm{r}} \\ \mathbf{z}_{\mathrm{fd}}^{\mathrm{b}} \\ \mathbf{z}_{\mathrm{fr}}^{\mathrm{i}} \\ \mathbf{z}_{\mathrm{fr}}^{\mathrm{b}} \end{bmatrix} \tag{29}$$

Premultiplying $(\mathbf{I}-\mathbf{M}^{*}\mathbf{M}_{\mathrm{o}})^{-1}$ on both sides of equation (28) and using $\mathbf{u}_{\mathrm{p}}=\bar{\mathbf{u}}_{\mathrm{p}}e^{\lambda t}$, it follows that

$$\lfloor \lambda^{2}+\lambda\mathbf{C}^{*}+\mathbf{K}^{*}\rfloor\,\bar{\mathbf{u}}_{\mathrm{p}}e^{\lambda t}=\mathbf{0} \tag{30}$$

where $\mathbf{C}^{*}=(\mathbf{I}-\mathbf{M}^{*}\mathbf{M}_{\mathrm{o}})^{-1}\mathbf{M}^{*}\mathbf{C}$ and $\mathbf{K}^{*}=(\mathbf{I}-\mathbf{M}^{*}\mathbf{M}_{\mathrm{o}})^{-1}\mathbf{M}^{*}\mathbf{K}$. The eigenvalues can be readily solved by using *roots* in MATLAB (1992).

Also, the eigenvalues are obtained by using a state-space approach, which replaces $(g+l+2h)$ coupled second order differential equations by $2(g+l+2h)$ coupled first order differential equations. Introducing a state vector of length $2(g+l+2h)$

$$\eta=[\dot{\mathbf{u}}_{p}^{T}\ \mathbf{u}_{p}^{T}]^{T} \tag{31}$$

into equation (28), it can be rewritten in a form that consists of $2(g+l+2h)$ simultaneous first order ordinary differential equations as

$$\mathbf{W}\dot{\eta}-\mathbf{R}\eta=0 \tag{32}$$

where matrices $\mathbf{W}$ and $\mathbf{R}$ are both symmetric and are given by

$$\mathbf{W}=\begin{bmatrix} 0 & \mathbf{I}-\mathbf{M}^{*}\mathbf{M}_{\mathrm{o}} \\ \mathbf{I}-\mathbf{M}^{*}\mathbf{M}_{\mathrm{o}} & \mathbf{M}^{*}\mathbf{C} \end{bmatrix}$$
$$\text{and }\mathbf{R}=\begin{bmatrix} \mathbf{I}-\mathbf{M}^{*}\mathbf{M}_{\mathrm{o}} & 0 \\ 0 & -\mathbf{M}^{*}\mathbf{K} \end{bmatrix} \tag{33}$$

Substitution of $\eta=\bar{\eta}e^{\lambda t}$ into equation (32) yields the $2(g+l+2h)\times 2(g+l+2h)$ generalized eigenvalue problem

$$\mathbf{R}\bar{\eta}=\lambda\mathbf{W}\bar{\eta} \tag{34}$$

where λ corresponds to the eigenvalue of the system. Equation (34) can be solved by using *eig* in MATLAB. The solutions of equations (30) and (34) represent the eigenvalues of the entire structure composed of the primary structure and the attached secondary systems.

5. Application (frequency response matrix)

Consider a vibrational linear system consisting of three masses m_{i}, $(i=1, 2, 3)$ connected by springs, and a viscous damper at the first mass, as shown in Fig. 2 (Gurgoze, 2000). Representing

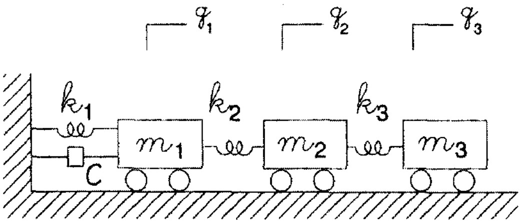

Fig. 2 A 3 DOF system with a single viscous damping

the displacements of the masses m_i by $q_i(t)$ $(i=1, 2, 3)$, the system matrices are expressed as

$$\mathbf{M}=\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \mathbf{C}=\begin{bmatrix} c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{K}=\begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \quad (35)$$

To evaluate the numerical results for frequency response matrix of the given system, the following values were selected ;

$$\begin{aligned} &k_1=k_2=10 \text{ N/m}, \ k_3=20\text{N/m} \\ &m_1=1 \text{ kg}, \ m_2=2 \text{ kg}, \ m_3=4 \text{ kg}, \\ &c=10 \text{ N/m/s}, \ \omega=2 \text{ rad/sec} \end{aligned} \quad (36)$$

The receptance matrix of the unconstrained system by equation (13) and MATLAB was calculated as

$$\mathbf{H}=\begin{bmatrix} 0.0247-0.0286i & -0.0032+0.0037i & -0.0159+0.0183i \\ -0.0032+0.0037i & -0.0124-0.0005i & -0.0621-0.0024i \\ -0.0159+0.0183i & -0.0621-0.0024i & -0.0603-0.0118i \end{bmatrix} \quad (37)$$

Assuming that the system is subjected to a constraint

$$q_1+2q_3=q_2 \quad (38)$$

the coefficient matrices $\mathbf{A}$ and $\mathbf{b}$ of equation (14) can be written as

$$\mathbf{A}=[1 \ -1 \ 2], \ \mathbf{b}=[0 \ 0 \ 0] \quad (39)$$

Consequently, the frequency response function of the constrained system was obtained by equation (15) and MATLAB, and the coefficient matrices and the frequency response matrix give that

$$\mathbf{C}=\begin{bmatrix} -6.0-12.0i & -4.0 & 8.0 \\ -8.0i & -8.0 & 12.0 \\ 20.0+16.0i & -8 & 12.0 \end{bmatrix}, \mathbf{D}=\begin{bmatrix} -0.6 & -0.2 & 0.2 \\ -0.4 & -0.8 & -0.2 \\ 0.8 & -0.4 & -0.6 \end{bmatrix}$$
$$\hat{\mathbf{H}}=\begin{bmatrix} 0.0246-0.0295i & 0.0082-0.0098i & -0.0082+0.0098i \\ 0.0082-0.0098i & 0.3361-0.0033i & 0.1639+0.0033i \\ -0.0082+0.0098i & 0.1639+0.0033i & 0.0861-0.0033i \end{bmatrix} \quad (40)$$

The final result $\hat{\mathbf{H}}$ coincides with that obtained from the equation provided by Gurgoze (1999).

As another application, let us assume that the above-unconstrained system is subjected to a constraint

$$q_1+2q_3=[1 \ 1 \ 3]\bar{\mathbf{F}}e^{i\omega t} \quad (41)$$

It can be shown that the above constraint is a nonhomogeneous function. This kind of constraint cannot be handled in the method provided by Gurgoze. Differentiating equation (41) twice with respect to time t, it can be written in matrix form

$$-\omega^2[1 \ 0 \ 2]\begin{bmatrix} q_1 e^{i\omega t} \\ q_2 e^{i\omega t} \\ q_3 e^{i\omega t} \end{bmatrix} = -\omega^2[1 \ 1 \ 3]\bar{\mathbf{F}}e^{i\omega t} \quad (42)$$

Using the coefficient matrices of equations (35) and (42) into equation (15), the frequency response matrix is obtained as

$$\mathbf{C}=\begin{bmatrix} -9.8462-13.8462i & 10.0 & 18.4615+18.4615i \\ 2.3077 & -22.0 & -3.0769 \\ 4.6154i & 20.0 & 9.8462 \end{bmatrix}$$
$$\mathbf{D}=\begin{bmatrix} 0.5385 & 1.2308 & 1.4615 \\ 0 & -1.0 & 0 \\ 4.6154 & 3.6923 & 3.3846 \end{bmatrix} \quad (43)$$
$$\hat{\mathbf{H}}=\begin{bmatrix} 0.0598-0.0751i & 0.0391-0.0491i & 0.0273-0.0343i \\ 0.3121-0.0114i & 0.3544-0.0074i & 0.3072-0.0052i \\ 0.3134+0.0250i & 0.3203+0.0164i & 0.3242+0.0114i \end{bmatrix}$$

Through these applications, it is verified that the proposed method takes simpler and more general form than the Gurgoze's method.

6. Longitudinally Vibrating Rod

The partial differential equation of longitudinal vibration of a tapered rod fixed at $x=0$ and free at $x=L$ shown in Fig. 3(a) can be written by

$$EA(x)\,u''(x, t)=m(x)\,\ddot{u}(x, t) \quad (44)$$

where $u(x, t)$ represents the axial displacement of the beam at point x and time t. The primes and overdots denote partial derivatives with respect to x and t, respectively. Let us assume that the axial displacement $u(x, t)$ is separable in space and time, or

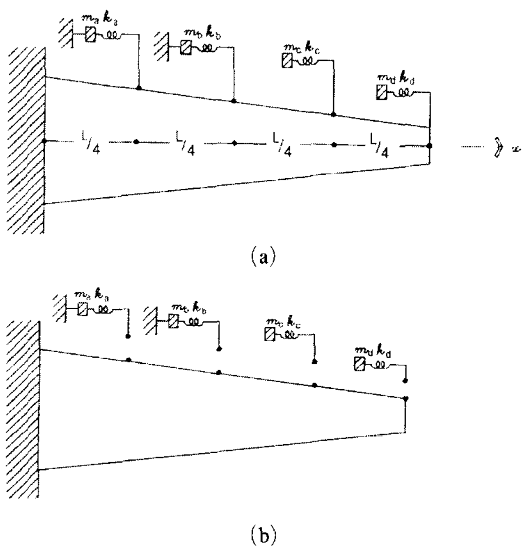

Fig. 3 A continuous system of tapered rod ; (a) a entire system composed of a tapered rod and discrete systems, (b) decomposition of a tapered rod and discrete systems

$$u(x,\ t)=\sum_{i=1}^{n}\phi_i(x)\,q_i(t) \tag{45}$$

where $\phi_i(x)$ are the eigenfunctions of the linear structure, $q_i(t)$ are the corresponding generalized coordinates, and n is the number of modes used in the assumed-modes expansion.

Utilizing equation (45) into the kinetic and potential energies of the system, they can be formulated as

$$\mathrm{T}=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}m_{ij}\dot{q}_i(t)\,\dot{q}_j(t) \tag{46a}$$

$$\mathrm{V}=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}k_{ij}q_i(t)\,q_j(t) \tag{46b}$$

where

$$m_{ij}=\int_0^L \mathrm{m}(x)\,\phi_i(x)\,\phi_j(x)\,dx \tag{47a}$$

$$k_{ij}=\int_0^L \mathrm{EA}(x)\frac{d\phi_i(x)}{dx}\,\frac{d\phi_j(x)}{dx}\,dx \tag{47b}$$

Substituting equations (46) into Lagrange's equations, it follows that

$$\begin{aligned}&\frac{d}{dt}\left(\frac{\partial \mathrm{T}}{\partial \dot{q}_r}\right)-\frac{\partial \mathrm{T}}{\partial q_r}+\frac{\partial \mathrm{V}}{\partial q_r}\\&=\sum_{j=1}^{n}m_{rj}\ddot{q}_j(t)+\sum_{j=1}^{n}k_{rj}q_j(t)=0,\ r=1,\ 2,\ \cdots,\ n\end{aligned} \tag{48}$$

Expressing equation (48) in matrix form, the longitudinal vibration equation of the tapered rod is written as

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{0} \tag{49}$$

Assuming that $(l+h)$ secondary systems composed of l fixed-free system and h free-free system are attached to the primary system, the vibration equations of the secondary systems are written in the same form as equation (17). The compatibility conditions at junctions between the rod and the attached secondary systems are expressed as equations (24).

The stiffness and mass distributions of the given rod are

$$\begin{aligned}EA(x)&=\frac{6}{5}\,EA\left[1-\frac{1}{2}\left(\frac{x}{L}\right)^2\right]\\ m(x)&=\frac{6}{5}\,m\left[1-\frac{1}{2}\left(\frac{x}{L}\right)^2\right]\end{aligned} \tag{50}$$

respectively. The eigenfunctions corresponding to a uniform rod clamped at $x=0$ and free at $x=L$ were assumed as

$$\phi_i(x)=\sin(2i-1)\frac{\pi x}{2L},\ i=1,\ 2,\ \cdots,\ n \tag{51}$$

Substituting equations (50) and (51) into equations (47a) and (47b), and utilizing the result into equation (49), the equation of motion of the undamped discretized rod is obtained. Assume that four substructures are attached at locations indicated in Fig. 3(a). Two substructures of them have fixed end and the others have free end. If the substructures are separated from the primary rod as shown in Fig. 3(b), the equations of motion of the substructures are written in the form of equation (17). Additionally assume that the longitudinal vibration of the rod is constrained by the linear relation of

$$\begin{aligned}&2u(x=L/4,\ t)-5u(x=L/2,\ t)\\&-4u(x=3L/4,\ t)+3u(x=L,\ t)=0\end{aligned} \tag{52}$$

It is observed that the constraint equation (52) establishes rigid-body motion in spite of the presence of subsystems. Using equation (45) into equation (52), the coefficient matrix $\mathbf{A}$ of equation (27) by the constraint and compatibility conditions should be written as

$$\mathbf{A}=\begin{bmatrix} \phi_{12} & \phi_{22} & \phi_{32} & \phi_{42} & -1 & 0 & 0 & 0 & 0 & 0 \\ \phi_{14} & \phi_{24} & \phi_{34} & \phi_{44} & 0 & -1 & 0 & 0 & 0 & 0 \\ \phi_{16} & \phi_{26} & \phi_{36} & \phi_{46} & 0 & 0 & 0 & 0 & -1 & 0 \\ \phi_{18} & \phi_{28} & \phi_{38} & \phi_{48} & 0 & 0 & 0 & 0 & 0 & -1 \\ s1 & s2 & s3 & s4 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (53)$$

where $s1=2\phi_{12}-5\phi_{22}-4\phi_{32}+3\phi_{42}$, $s2=2\phi_{14}-5\phi_{24}-4\phi_{34}+3\phi_{44}$, $s3=2\phi_{16}-5\phi_{26}-4\phi_{36}+3\phi_{46}$, $s4=2\phi_{18}-5\phi_{28}-4\phi_{38}+3\phi_{48}$, and the subscripts i and j of ϕ_{ij} indicate the eigenvector corresponding to the jth modal displacement of the ith mode.

Utilizing the numerical values of $m_a=m_b=m_c=m_d=1$, $L=200$, $E=2\times10^5$, $A=30$, $n=8$, $m=1$, $k_a=3000$, $k_b=9000$, $k_c=7000$ and $k_d=3000$, the lowest frequencies were calculated by using equations (30) or (34), and MATLAB version 5.1 on a PC Pentium III. The results are shown in Table 1. The table represents the eigenfrequency of the entire structure with and without the additional constraint (52). The presence of the additional constraint yields the rigid-body motion of the entire structure, hence the lowest eigenfrequency must be zero.

This application illustrates the effectiveness and easiness to calculate the eigenvalues of complicated continuous or discrete structures subjected to multiple linear constraints including compatibility conditions. Thus, the proposed method will be extensively utilized to the substructure synthesis and other eigenvalue problems.

Table 1 Eigenvalues of the entire structure with and without the additional constraint

Mode number	Eigenvalues (rad./sec.)	
	Without additional constraint (52)	With additional constraint (52)
1^{st}	21.7	0
2^{nd}	54.8	21.7
3^{rd}	59.0	54.8
4^{th}	83.7	74.2
5^{th}	97.1	83.7
6^{th}	135.0	97.1

7. Conclusions

Starting from the generalized inverse method provided by Udwadia and Kalaba, this study determined the frequency response matrix of constrained system subjected to harmonic forces and multiple constraints. The derived frequency response matrix took a more generalized form than other results. Although the generalized inverse method cannot handle the constrained systems to have positive semidefinite mass matrix, this study extended it to the systems of positive semidefinite mass matrix and the continuous systems, and determined the dynamic characteristics. It was shown that the proposed approach is effective for establishing the dynamic characteristics of complicated systems composed of various subsystems as well as continuous systems subjected to constraints.

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